



On the triplet vertex algebra $\mathcal{W}(p)$

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Abstract

We study the triplet vertex operator algebra $\mathcal{W}(p)$ of central charge $1 - \frac{6(p-1)^2}{p}$, $p \geq 2$. We show that $\mathcal{W}(p)$ is C_2 -cofinite but irrational since it admits indecomposable and logarithmic modules. Furthermore, we prove that $\mathcal{W}(p)$ is of finite-representation type and we provide an explicit construction and classification of all irreducible $\mathcal{W}(p)$ -modules and describe block decomposition of the category of ordinary $\mathcal{W}(p)$ -modules. All this is done through an extensive use of Zhu's associative algebra together with explicit methods based on vertex operators and the theory of automorphic forms. Moreover, we obtain an upper bound for $\dim(A(\mathcal{W}(p)))$. Finally, for p prime, we completely describe the structure of $A(\mathcal{W}(p))$. The methods of this paper are easily extendable to other \mathcal{W} -algebras and superalgebras.

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0. Introduction

The main focus of vertex operator algebra theory so far has been on understanding *rational vertex operator algebras*. This progress has led, in particular, to several important breakthroughs in the area such as Zhu's modular invariance theorem [42] and Huang's recent proof of the Verlinde's conjecture [30]. All these developments are deeply rooted in ideas of conformal field theory.

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On the other hand, irrational vertex operator algebras did not attract so much attention for several obvious reasons. The category of modules of a (irrational) vertex operator algebra has often too many irreducible objects, which forces many important features such as modular invariance to be absent. In view of that, it is reasonable to focus first on vertex algebras of *finite-representation type* and relax the semisimplicity condition. A vertex operator algebra V will be of finite-representation type if its Zhu associative algebra is finite-dimensional, which is the case when V is C_2 -cofinite (cf. [12]). Since the C_2 -cofiniteness property is rather strong, it is interesting that there are examples of irrational C_2 -cofinite vertex operator algebras (cf. [1]). More surprisingly, there is a version of modular invariance theorem for modules of C_2 -cofinite vertex algebras [41], so irrational C_2 -cofinite vertex algebras appear to be very special objects. Since there is only a handful of known examples of vertex algebras with these properties, it is important to study the known examples and to seek for new models.

Our present line of work, continuing [5], is concerned with “ \mathcal{W} -algebras.” The \mathcal{W} -algebras have been studied intensively by the mathematicians and physicists over the last two decades. These “algebras” are not Lie algebras in the classical sense, but rather close cousins of vertex algebras associated to affine Lie algebras and lattice vertex algebras, and are usually defined via the process of *quantum reduction* [23,24]. The \mathcal{W} -algebras and their classical counterparts are important not only *per se*, but also in connection with integrable hierarchies, opers and even geometric Langlands correspondence [23]. For some recent advances in the theory of \mathcal{W} -algebras see [7,9,22], etc.

The \mathcal{W} -algebras associated to affine Lie algebras come in families parameterized by the value of central charge. For generic values of the parameter, \mathcal{W} -algebras have fairly explicit description and are known to be irrational (cf. [7,24]). For non-generic values the situation seems to be opposite. The simplest examples of non-generic \mathcal{W} -algebras—the vacuum Virasoro minimal models—are known to be rational. But in higher “rank,” there are almost no classification results of representations of non-generic \mathcal{W} -algebras. A sole example would be the proof of rationality of a rank two \mathcal{W}_3 -algebra of central charge $c = \frac{6}{5}$, which is already quite involved [11]. Even so, one does expect that, suitably defined, vacuum minimal models for higher rank \mathcal{W} -algebras are also rational vertex algebras [23].

In this paper we focus on a prominent family of \mathcal{W} -algebras—called *triplet \mathcal{W} -algebras*—introduced by Kausch in [34,35]. These are parameterized by a positive integer $p \geq 2$, and have central charge $c = 1 - \frac{6(p-1)^2}{p}$, $p \geq 2$. Unlike the \mathcal{W} -algebras discussed in the previous paragraph, the triplet is associated to a (non-root) lattice vertex algebra. In terms of generators the triplet comes with four distinguished vectors; one generator is the usual Virasoro vector and the remaining three are certain primary fields of conformal weight $2p - 1$. For $p = 2$, the triplet has central charge -2 and it admits a realization via the so-called *symplectic fermions* [27,28,34]. A great deal of research has been done on the symplectic fermions from several different points of view. One particularly interesting feature of this model is the appearance of the so-called *logarithmic modules* (i.e., modules with a non-diagonalizable action of the Virasoro generator $L(0)$). This links the $c = -2$ model and more general triplets with *logarithmic conformal field theory* (LCFT), a new physical theory with possible applications in condensed matter physics and string theory (cf. see [31,32] for a vertex operator algebra approach to LCFT). There is a large body of work devoted to various interactions between the triplet \mathcal{W} -algebras, LCFT and the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ at a root of unity. For these and other related developments we refer the reader to [13,15–19,21,25,27–29,34–36], and especially two excellent reviews: [20] and [26].

Since the triplet \mathcal{W} -algebras are in fact vertex (operator) algebras, it is tempting to use vertex algebra theory to analyze these objects. In [1] Abe studied the $c = -2$ triplet vertex algebra,

denoted by $\mathcal{W}(2)$, by using vertex algebra theory. He eventually succeeded in classifying irreducible $\mathcal{W}(2)$ -modules and describing the Zhu's associative algebra $A(\mathcal{W}(2))$. A rather explicit fermionic construction of $\mathcal{W}(2)$ played a prominent role in Abe's approach. However, for $p > 2$ the triplet vertex algebra $\mathcal{W}(p)$ has no natural fermionic realization so one needs a completely different approach (cf. [25]) to study these vertex algebras and to extend classification results to $\mathcal{W}(p)$, for $p > 2$. That was precisely the main motivational problem for this paper.

Here is a short description of our main results. We start with a rank one lattice vertex algebra V_L and view the triplet $\mathcal{W}(p)$ as the kernel of a screening operator acting on V_L . It is important to say here that L is not a root lattice. Alternatively, the triplet can be defined in terms of generators as we indicated earlier. Our first two results are the classification of irreducible $\mathcal{W}(p)$ -modules and a fairly simple proof of the C_2 -cofiniteness:

Theorem A.

- (i) *The triplet vertex algebra $\mathcal{W}(p)$ is C_2 -cofinite.*
- (ii) *The triplet vertex algebra $\mathcal{W}(p)$ has precisely $2p$ inequivalent irreducible modules.*
- (iii) *The vertex algebra $\mathcal{W}(p)$ is irrational.*

For precise statements see Theorems 2.1, 3.7 and 3.12. One should say that Theorem A is in agreement with the results obtained by physicists. We also stress that the C_2 -cofiniteness part (i) was discussed in [8] by using a completely different approach.

Since the triplet vertex algebra is not rational (cf. Proposition 4.2) it is not evident how to obtain a decomposition of Zhu's associative algebra $A(\mathcal{W}(p))$ as a direct sum of its ideals. Even though Theorem A indicates that $A(\mathcal{W}(p))$ should be a sum of $2p$ ideals, it is not clear what precisely are those ideals and how to compute their dimensions. Our next result gives a partial answer to this problem.

Theorem B. *Zhu's algebra $A(\mathcal{W}(p))$ decomposes as a direct sum of ideals*

$$A(\mathcal{W}(p)) = \bigoplus_{i=2p}^{3p-1} \mathbb{M}_{h_{i,1}} \oplus \bigoplus_{i=1}^{p-1} \mathbb{I}_{h_{i,1}} \oplus \mathbb{C},$$

where $\mathbb{M}_{h_{i,1}}$ is an ideal isomorphic to $M_2(\mathbb{C})$, and each $\mathbb{I}_{h_{i,1}}$ is at most two-dimensional (all ideals are parameterized by certain conformal weights $h_{i,1}$).

In fact, we prove much more; all our ideals are described with explicit spanning sets (bases?), which is useful for computational purposes.

Notice that the last result is not completely satisfactory because we really do not know whether $\mathbb{I}_{h_{i,1}}$ is two-dimensional (this is our conjecture though). The problem relies on the existence of certain logarithmic $\mathcal{W}(p)$ -modules, predicted by physicists (cf. [25]). In the $p = 2$ case the existence of such a module can be easily seen by using symplectic fermions, but for $p > 2$ it is not quite clear how to construct these modules explicitly. Nevertheless, by using a work of Miyamoto [41] we settle this problem, at least when p is a prime integer.

Theorem C. *Suppose that p is a prime number. Then each $\mathbb{I}_{h_{i,1}}$ is two-dimensional and*

$$\dim(A(\mathcal{W}(p))) = 6p - 1.$$

For general p we only have a partial result in this direction (cf. Proposition 6.6).

Besides an obvious representation theoretic interest, our results and techniques have other merits. For example, methods used in this paper are applicable to other \mathcal{W} -algebras and super-algebras defined via screening operators. This paper also gives rigorous proofs of several claims in physics literature about the triplet, their representations and logarithmic modules. Thus, our results have immediate applications in logarithmic conformal field theory.

1. The triplet vertex algebra $\mathcal{W}(p)$

In this section we introduce the triplet vertex algebra and study some of its representations.

The setup is similar as in [5] (see also [4]) so we omit many details. For an integer $p \geq 2$, we fix a rank one lattice $\mathbb{Z}\alpha$, generated by α with

$$\langle \alpha, \alpha \rangle = 2p.$$

We denote by $(V_L, Y, \omega, \mathbf{1})$ the corresponding lattice vertex operator algebra [10,37]. As a vector space,

$$V_L = \mathcal{U}(\hat{\mathfrak{h}}_{<0}) \otimes \mathbb{C}[L],$$

where $\mathbb{C}[L]$ is the group algebra of L and $\hat{\mathfrak{h}}$ is the affinization of the one-dimensional algebra spanned by α , the vacuum vector

$$\mathbf{1} = 1 \otimes 1$$

and the conformal vector

$$\omega = \frac{\alpha(-1)^2}{4p} \mathbf{1} + \frac{p-1}{2p} \alpha(-2) \mathbf{1}.$$

Notice that this is not the usual quadratic Virasoro generator used throughout the literature. For convenience, let us fix $h = \frac{\alpha}{\sqrt{2p}}$, so that $\langle h, h \rangle = 1$.

It is known that the vertex operator algebra V_L is rational, in the sense that it has finitely many irreducible V_L -modules and that every V_L -module is completely reducible (see [37] for instance). If we denote by $L^\circ = \frac{\mathbb{Z}\alpha}{2p}$ the dual lattice of L , then

$$V_{L+\lambda} = \mathcal{U}(\hat{\mathfrak{h}}_{<0}) \otimes e^\lambda \mathbb{C}[L], \quad \lambda \in L^\circ/L,$$

are, up to equivalence, all irreducible V_L -modules. For $\lambda = 0$ we recover the vertex algebra V_L , while V_{L° has a generalized vertex operator algebra structure. The Virasoro algebra acts on $V_{L+\lambda}$ with the central charge

$$c_{p,1} = 1 - \frac{6(p-1)^2}{p}.$$

As usual the Virasoro generators will be denoted by $L(n)$, $n \in \mathbb{Z}$. Then the degree zero operators $L(0)$ and the charge operator $h(0)$ equip $V_{L+\lambda}$ with a compatible bigrading.

We also define rational numbers

$$h_{m,n} = \frac{(m - np - p + 1)(m - np + p - 1)}{4p}, \quad m, n \in \mathbb{Z},$$

parameterizing $(1, p)$ -minimal models at the boundary of Kac's table. In fact, it is sufficient to consider the weights

$$h_{m+1,1} = \frac{m(m - 2p + 2)}{4p}, \quad m \geq 0. \quad (1.1)$$

As in [4] and [5] we let

$$\begin{aligned} Q &: V_L \longrightarrow V_L, \\ \tilde{Q} &: V_L \longrightarrow V_{L^\circ}, \end{aligned}$$

where

$$\begin{aligned} Q &= e_0^\alpha, \\ \tilde{Q} &= e_0^{-\alpha/p} \end{aligned}$$

are *screening* operators introduced by Dotsenko and Fateev (cf. [23]). As usual, here

$$Y(e^\alpha, x) = \sum_{n \in \mathbb{Z}} e_n^\alpha x^{-n-1}, \quad Y(e^{-\alpha/p}, x) = \sum_{r \in \frac{1}{p}\mathbb{Z}} e_r^{-\alpha/p} x^{-r-1},$$

where the second vertex operator belongs to the generalized vertex algebra V_{L° . We stress here that the screening operator Q acts as a derivation

$$Q(a_nb) = (Qa)_nb + a_n(Qb), \quad a, b \in V_L, \quad n \in \mathbb{Z}.$$

This formula will be extensively used throughout the paper.

Inside the vertex algebra V_L we consider the following three vectors and their vertex operators, crucial for our present work:

$$\begin{aligned} F &= e^{-\alpha}, \\ H &= Qe^{-\alpha}, \\ E &= Q^2e^{-\alpha}, \end{aligned}$$

$$Y(X, z) = \sum_{i \in \mathbb{Z}} X_i z^{-i-1}, \quad X \in \{E, F, H\}.$$

As we shall see later, the choice of letters F , H and E resembling the standard \mathfrak{sl}_2 generators is not an accident.

Let us recall the formulas (cf. [4,5]):

$$E_i E = F_i F = Q(H_i H) = 0, \quad i \geq -2p; \quad (1.2)$$

$$[Q, \tilde{Q}] = 0. \quad (1.3)$$

We shall need the following useful formulae which hold in V_L

$$Y(e^\alpha, x_1)Y(e^\alpha, x_2) = E^-(\alpha, x_1, x_2)E^+(\alpha, x_1, x_2)(x_1 - x_2)^{2p}e^{2\alpha}(x_1 x_2)^\alpha \quad (1.4)$$

where

$$E^\pm(\alpha, x_1, x_2) = \exp\left(\sum_{k=1}^{\infty} \frac{\alpha(\pm k)}{\pm k} (x_1^{\mp k} + x_2^{\mp k})\right).$$

For $i \in \mathbb{Z}$, we set

$$\gamma_i = \frac{i}{2p}\alpha. \quad (1.5)$$

We shall first present results on the structure of V_L -modules as modules for the Virasoro algebra. By using Lemma 4.3 from [5] and the structure theory of Feigin–Fuchs modules [14] we get the following theorems. (Here we also use that every V_L -module is a direct sum of Feigin–Fuchs modules due to charge decomposition.)

Theorem 1.1. Assume that $i \in \{0, \dots, p-2\}$.

- (i) As a Virasoro algebra module, $V_{L+\gamma_i}$ is generated by the family of singular and cosingular vectors $\widetilde{\text{Sing}}_i \cup \widetilde{\text{CSing}}_i$, where

$$\begin{aligned} \widetilde{\text{Sing}}_i &= \{u_i^{(j,n)} \mid j, n \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq 2n\}; \\ \widetilde{\text{CSing}}_i &= \{w_i^{(j,n)} \mid n \in \mathbb{Z}_{>0}, 0 \leq j \leq 2n+1\}. \end{aligned}$$

These vectors satisfy the following relations:

$$u_i^{(j,n)} = Q^j e^{\gamma_i - n\alpha}, \quad Q^j w_i^{(j,n)} = e^{\gamma_i + n\alpha}.$$

The submodule generated by singular vectors $\widetilde{\text{Sing}}_i$ is isomorphic to

$$\overline{V_{L+\gamma_i}} \cong \bigoplus_{n=0}^{\infty} (2n+1)L(c_{p,1}, h_{i+1,2n+1}).$$

- (ii) The quotient module is isomorphic to

$$V_{L+\gamma_i} / \overline{V_{L+\gamma_i}} \cong \bigoplus_{n=1}^{\infty} (2n)L(c_{p,1}, h_{i+1,-2n+1}).$$

(iii) As a Virasoro algebra module $V_{L+\gamma_{p-1}}$ is generated by the family of singular vectors

$$\widetilde{\text{Sing}}_{p-1} = \{u_{p-1}^{(j,n)} := Q^j e^{\gamma_{p-1}-n\alpha} \mid j, n \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq 2n\};$$

and it is isomorphic to

$$V_{L+\gamma_{p-1}} \cong \bigoplus_{n=0}^{\infty} (2n+1)L(c_{p,1}, h_{p,2n+1}).$$

Theorem 1.2. Assume that $i \in \{p, \dots, 2p-2\}$.

(i) As a Virasoro algebra module, $V_{L+\gamma_i}$ is generated by the family of singular and cosingular vectors $\widetilde{\text{Sing}}_i \cup \widetilde{\text{CSing}}_i$, where

$$\begin{aligned} \widetilde{\text{Sing}}_i &= \{u_i^{(j,n)} \mid j \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}, 0 \leq j \leq 2n-1\}; \\ \widetilde{\text{CSing}}_i &= \{w_i^{(j,n)} \mid j, n \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq 2n\}. \end{aligned}$$

These vectors satisfy the following relations:

$$u_i^{(j,n)} = Q^j e^{\gamma_i-n\alpha}, \quad Q^j w_i^{(j,n)} = e^{\gamma_i+n\alpha}.$$

The submodule generated by singular vectors $\widetilde{\text{Sing}}_i$ is isomorphic to

$$\overline{V_{L+\gamma_i}} \cong \bigoplus_{n=1}^{\infty} (2n)L(c_{p,1}, h_{i+1,2n+1}).$$

(ii) The quotient module is isomorphic to

$$V_{L+\gamma_i} / \overline{V_{L+\gamma_i}} \cong \bigoplus_{n=0}^{\infty} (2n+1)L(c_{p,1}, h_{i+1,-2n+1}).$$

(iii) As a Virasoro algebra module $V_{L+\gamma_{2p-1}}$ is generated by the family of singular vectors

$$\widetilde{\text{Sing}}_{2p-1} = \{u_{2p-1}^{(j,n)} := Q^j e^{\gamma_{2p-1}-n\alpha} \mid n \in \mathbb{Z}_{>0}, j \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq 2n-1\};$$

and it is isomorphic to

$$V_{L+\gamma_{2p-1}} \cong \bigoplus_{n=1}^{\infty} (2n)L(c_{p,1}, h_{2p,2n+1}).$$

We will be concerned with certain (vertex) subalgebras of the lattice vertex algebra V_L . For these purposes we recall a few basic definitions. Let V be a vertex (operator) algebra and $S \subset V$. Then we denote by $\langle S \rangle$ the vertex (operator) subalgebra generated by S (i.e., the smallest vertex

(operator) algebra containing the set S). Similarly if W is a subset of a V -module M , we denote by $\langle W \rangle$ the submodule of M generated by W .

The vertex (operator) algebra $\langle S \rangle$ is said to be *strongly generated* (cf. [33]) by S if it is spanned by vectors of the form

$$(v_1)_{n_1}(v_2)_{n_2} \cdots (v_k)_{n_k} \mathbf{1}, \quad v_i \in S, \quad n_i < 0.$$

Now, we are ready to introduce the main protagonist of the paper. For p as above we denote by $\mathcal{W}(p) \subset V_L$ the following vertex operator algebra [25]

$$\mathcal{W}(p) := \text{Ker}|_{V_L} \tilde{Q}.$$

Notice that this resembles the definition of the singlet vertex algebra $\overline{M(1)_p}$ studied in [4] and [5], with the only difference that now the kernel is taken over the whole lattice algebra, whereas the singlet vertex algebra is the kernel of \tilde{Q} restricted onto the charge zero subalgebra $M(1) \subset V_L$. We call $\mathcal{W}(p)$ the *triplet vertex algebra*. Clearly, $\overline{M(1)_p}$ is a vertex subalgebra of $\mathcal{W}(p)$. Recall also that $\overline{M(1)_p}$ is a simple vertex operator algebra strongly generated by ω and H (cf. [4,5]). Next result will identify the generators for $\mathcal{W}(p)$.

Proposition 1.3.

- (i) We have $\mathcal{W}(p) = \overline{V_L}$.
- (ii) The vertex operator (sub)algebra $\mathcal{W}(p) \subset V_L$ is strongly generated by E, H, F and ω .

Proof. Recall the structure of V_L as a module for the Virasoro algebra from Theorem 1.1. By using (1.3), similarly to the proof of Theorem 3.1 in [4], we conclude that $\mathcal{W}(p)$ is a completely reducible module for the Virasoro algebra which is generated by the family of singular vectors:

$$Q^j e^{-n\alpha}, \quad n \in \mathbb{Z}_{\geq 0}, \quad j \in \{0, \dots, 2n\}. \quad (1.6)$$

This proves (i). Let W_n be the Virasoro module generated by singular vectors

$$Q^j e^{-m\alpha}, \quad m \leq n, \quad j \in \mathbb{Z}_{\geq 0}.$$

Therefore $\mathcal{W}(p) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} W_n$. Let now U be the subspace of $\mathcal{W}(p)$ spanned by vectors of the form

$$(v_1)_{n_1}(v_2)_{n_2} \cdots (v_k)_{n_k} \mathbf{1}, \quad v_i \in \{E, F, H, \omega\}, \quad n_i < 0.$$

Clearly, $U \subseteq \mathcal{W}(p)$. We shall prove that $U = \mathcal{W}(p)$, i.e., $\mathcal{W}(p)$ is strongly generated by E, F, H, ω .

In order to prove that $U = \mathcal{W}(p)$ it is enough to show that $W_n \subseteq U$ for every $n \in \mathbb{Z}_{>0}$. We shall prove this claim by induction on n . By the definition, the claim holds for $n = 1$. Assume now that $W_n \subseteq U$. Set $j_0 = 2np + 1$. The results from [4] and [5] imply that

$$\begin{aligned} F_{-j_0} e^{-n\alpha} &= e^{-(n+1)\alpha}, \\ E_{-j_0} Q^{2n} e^{-n\alpha} &= C_{2n+1} Q^{2n+2} e^{-(n+1)\alpha}, \end{aligned}$$

where $C_{2n+1} \neq 0$ and

$$H_{-j_0} Q^j e^{-n\alpha} = C_j Q^{j+1} e^{-(n+1)\alpha} + v'_j,$$

where $v'_j \in W_n$, $C_j \neq 0$, $0 \leq j \leq 2n$. These relations imply that $W_{n+1} \subseteq U$. By induction we conclude that $W_n \subseteq U$ for every $n \in \mathbb{Z}_{>0}$ and therefore $U = \mathcal{W}(p)$. \square

It is not hard to see that in fact

Corollary 1.4. *The triplet vertex algebra $\mathcal{W}(p)$ is spanned by*

$$L(-r_1) \cdots L(-r_k) H_{-s_1} \cdots H_{-s_l} X_{-t_1} \cdots X_{-t_m} \mathbf{1},$$

where $X = E$ or $X = F$, $k, l, m \in \mathbb{N}$, $s_i \geq 1$, $r_i \geq 2$ and $t_i \geq 1$.

Remark 1. It is clear that $\mathcal{W}(p)$ can be defined for $p = 1$ as well. However, it is not hard to see that $\mathcal{W}(1)$ is the Virasoro vertex operator algebra $L(1, 0)$, while the vertex algebra generated by E , F and H and ω is all of V_L . In both cases we do not get a new vertex algebra.

2. C_2 -cofiniteness of $\mathcal{W}(p)$

As usual, for a vertex operator algebra V we let

$$C_2(V) = \{a_{-2}b : a, b \in V\}.$$

It is a fairly standard fact (cf. [42]) that $V/C_2(V)$ has a Poisson algebra structure with the multiplication

$$\bar{a} \cdot \bar{b} = \overline{a_{-1}b},$$

where $-$ denotes the natural projection from V to $V/C_2(V)$. If $V/C_2(V)$ is finite-dimensional we say that V is C_2 -cofinite.

The aim of this section is the following result.

Theorem 2.1. *The triplet VOA $\mathcal{W}(p)$ is C_2 -cofinite.*

Proof. Proposition 1.3 implies that $\mathcal{W}(p)/C_2(\mathcal{W}(p))$ is a commutative algebra generated by the set $\{\bar{E}, \bar{F}, \bar{H}, \bar{\omega}\}$. By using relation (1.2) and commutativity of $\mathcal{W}(p)/C_2(\mathcal{W}(p))$, we obtain that

$$\bar{E}^2 = \bar{F}^2 = 0.$$

Since

$$Q^2(F_{-1}F) = E_{-1}F + F_{-1}E + 2H_{-1}H = 0$$

we also have that

$$\bar{H}^2 = -\bar{E}\bar{F}$$

which implies that

$$\tilde{H}^4 = 0.$$

Moreover, the description of Zhu's algebra from [4] implies that

$$\tilde{H}^2 = C_p \tilde{\omega}^{2p-1} \quad (C_p \neq 0).$$

Since $\tilde{H}^4 = 0$, we conclude that $\tilde{\omega}^{4p-2} = 0$. Therefore, every generator of the commutative algebra $\mathcal{W}(p)/C_2(\mathcal{W}(p))$ is nilpotent and therefore $\mathcal{W}(p)/C_2(\mathcal{W}(p))$ is finite-dimensional. \square

Remark 2. We should say here that Theorem 2.1 was discussed by Carqueville and Flohr [8] by using a different circle of ideas. Their (rather lengthy) argument was based on analysis of characters.

3. Zhu's algebra $A(\mathcal{W}(p))$ and classification of irreducible $\mathcal{W}(p)$ -modules

In this part we classify all irreducible $\mathcal{W}(p)$ -modules. Our approach combines explicit methods (we will eventually realize all irreducible $\mathcal{W}(p)$ -modules inside the irreducible modules for the lattice vertex algebra V_L) together with an extensive use of Zhu's associative algebra [42]. Here and throughout the paper we shall assume some knowledge of the theory of C_2 -cofinite vertex algebras, in particular we will use results from [2] and references therein.

Let us recall some fairly standard notation and the definition of Zhu's algebra $A(V)$ associated to a vertex operator algebra V . As in [5] we shall always assume that

$$V = \coprod_{n \in \mathbb{Z}_{\geq 0}} V_n, \quad \text{where } V_n = \{a \in V \mid L(0)a = na\}.$$

For $a \in V_n$, we shall write $\text{wt}(a) = n$ or $\deg(a) = n$.

For a homogeneous element $a \in V$ we define the following bilinear maps $*$: $V \otimes V \rightarrow V$, \circ : $V \otimes V \rightarrow V$ as follows:

$$a * b := \text{Res}_x Y(a, x) \frac{(1+x)^{\text{wt}(a)}}{x} b = \sum_{i=0}^{\infty} \binom{\text{wt}(a)}{i} a_{i-1} b,$$

$$a \circ b := \text{Res}_x Y(a, x) \frac{(1+x)^{\text{wt}(a)}}{x^2} b = \sum_{i=0}^{\infty} \binom{\text{wt}(a)}{i} a_{i-2} b.$$

Next, we extend $*$ and \circ on $V \otimes V$ linearly, and denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The space $A(V)$ has a unitary associative algebra structure, the Zhu's algebra of V . The image of $v \in V$, under the natural map $V \mapsto A(V)$ will be denoted by $[v]$.

Assume that $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ is a $\mathbb{Z}_{\geq 0}$ -graded V -module. Then the top component $M(0)$ of M is a $A(V)$ -module under the action $[a] \mapsto o(a) = a_{\text{wt}(a)-1}$ for homogeneous a in V . Moreover, there is one-to-one correspondence between irreducible $A(V)$ -modules and irreducible $\mathbb{Z}_{\geq 0}$ -graded V -modules (cf. [42]).

Unlike the Poisson algebra $V/C_2(V)$ that inherits the V -grading, Zhu's associative algebra $A(V)$ is not graded, but it does admit an increasing filtration. The corresponding (commutative) associative graded algebra is then denoted by $\text{gr}_\bullet(A(V))$ (see [1]). Then we have a natural epimorphism of commutative algebras

$$\pi : V/C_2(V) \longrightarrow \text{gr}_\bullet(A(V)).$$

This and the previous discussion give the following useful result (see [1] or [41] for details).

Proposition 3.1. *Let V be strongly generated by the set S . Then $A(V)$ is generated by the set*

$$\{[a], a \in S\}.$$

Moreover, if V is C_2 -cofinite

$$\dim(V/C_2(V)) \geq \dim(A(V)).$$

Now we specialize $V = \mathcal{W}(p)$. Our first goal is to gain some information about the Zhu's algebra (we will have to work more to obtain additional relations). First, we recall (cf. [42]) that for $a \in V$ homogeneous

$$\text{Res}_x Y(a, x)b \frac{(x+1)^{\text{wt}(a)}}{x^{2+n}} \in O(\mathcal{W}(p)) \quad \forall a, b \in \mathcal{W}(p), n \geq 0. \quad (3.7)$$

This implies the following lemma:

Lemma 3.2. *We have*

$$Q^2 e^{-2\alpha} \in O(\mathcal{W}(p)).$$

Proof. From the relations

$$\begin{aligned} e_{-2p-1}^{-\alpha} e^{-\alpha} &= e^{-2\alpha}, \\ e_n^{-\alpha} e^{-\alpha} &= 0, \quad n \geq -2p, \end{aligned}$$

and (3.7) we obtain

$$e^{-2\alpha} \in O(\mathcal{W}(p)).$$

Since Q preserves $O(\mathcal{W}(p))$, the proof follows. \square

By using [4] and [5] one can obtain the following lemma.

Lemma 3.3. *We have*

$$\begin{aligned} H_{-2p-1}H &\in \mathcal{U}(\text{Vir}).Q^2 e^{-2\alpha} \oplus \mathcal{U}(\text{Vir}).\mathbf{1} \subset M(1), \\ H_{-2p-1}H &= C Q^2 e^{-2\alpha} + v', \end{aligned}$$

where $C \neq 0$ and $v' \in \mathcal{U}(\text{Vir}).\mathbf{1}$.

Proof. We already know (cf. [4,5]),

$$H_{-2p-1}H = CQ^2e^{-2\alpha} + v' + v'', \quad (3.8)$$

where C is a non-zero complex number, $v' \in \mathcal{U}(\text{Vir}).\mathbf{1}$ and $v'' \in \mathcal{U}(\text{Vir}).H$. (Remember that $Q^2e^{-2\alpha} = u_0^{2,2}$ is a highest weight vector for the Virasoro algebra.) Notice that each vector on the right-hand side of (3.8) is of conformal weight $6p - 2$. We also recall

$$H_i H \in \mathcal{U}(\text{Vir}).\mathbf{1}, \quad i \geq -2p.$$

Suppose that $v'' \neq 0$. The vector v'' cannot be singular so then either $L(1)v''$ or $L(2)v''$ are non-trivial. Suppose that $L(1)v'' \neq 0$. Then the relation

$$L(1)H_{-2p-1}H = aH_{-2p}H \in \mathcal{U}(\text{Vir}).\mathbf{1}, \quad \text{for some } a \in \mathbb{C},$$

together with (3.8) imply $L(1)v'' \in \mathcal{U}(\text{Vir}).\mathbf{1}$, contradicting $L(1)v'' \in \mathcal{U}(\text{Vir}).H$. The same way we treat the case $L(2)v'' \neq 0$. \square

The next lemma will give a very useful binomial identity.

Lemma 3.4. *We have the following identity inside $\mathbb{C}[t]$:*

$$\widetilde{\Phi}_p(t) = \sum_{i=0}^{2p} (-1)^i \binom{2p}{i} \binom{t}{4p-1-i} \binom{t}{2p+i-1} = A_p \binom{t}{3p-1} \binom{t+p}{3p-1}, \quad (3.9)$$

where $A_p = (-1)^p \frac{\binom{2p}{p}}{\binom{4p-1}{p}}$.

Proof. Let $g_p(t) = A_p \binom{t}{3p-1} \binom{t+p}{3p-1}$. It is clear that both $g_p(t)$ and $\widetilde{\Phi}_p(t)$ are inside $\mathbb{C}[t]$. First we notice that for $t = 3p - 1$ there is only one non-trivial term in (3.9), so

$$\widetilde{\Phi}_p(3p-1) = (-1)^p \binom{2p}{p} = A_p \binom{4p-1}{3p-1} = g_p(3p-1). \quad (3.10)$$

By using straightforward calculation we get the following recursion:

$$(t+1)(p+t+1)\widetilde{\Phi}_p(t) = (2p-t-2)(3p-t-2)\widetilde{\Phi}_p(t+1). \quad (3.11)$$

By using (3.10) and the fact that $g_p(t)$ also satisfies the same recursion we conclude that $\widetilde{\Phi}_p(t) = g_p(t)$ for infinitely values of $t \in \mathbb{C}$. The proof follows. \square

Theorem 3.5. *In Zhu's algebra $A(\mathcal{W}(p))$ we have the following relation*

$$f_p([\omega]) = 0,$$

where

$$f_p(x) = \prod_{i=0}^{3p-2} (x - h_{i+1,1}). \quad (3.12)$$

Proof. By using Lemma 3.3 we have that (in $A(\overline{M(1)_p})$, where $\overline{M(1)_p}$ is the singlet algebra)

$$[Q^2 e^{-2\alpha}] = \Phi_p([\omega]) \quad (3.13)$$

for certain $\Phi_p \in \mathbb{C}[x]$, $\deg \Phi_p \leq 3p - 1$. We will see that Φ_p is a non-trivial polynomial of degree $3p - 1$, and find explicit formulae for this polynomial. Since $Q^2 e^{-2\alpha} \in \overline{M(1)_p} \subset M(1)$, we shall evaluate the action of $Q^2 e^{-2\alpha}$ on top levels of $\overline{M(1)_p}$ -modules $M(1, \lambda)$. Let v_λ be the highest weight vector in $M(1, \lambda)$. We use the method of [3] and [4]. By using (1.4) and direct calculation we see that

$$\begin{aligned} o(Q^2 e^{-2\alpha})v_\lambda &= \Phi_p\left(\frac{1}{4p}(t^2 - 2t(p-1))\right)v_\lambda \\ &= \text{Res}_{z_1} \text{Res}_{z_2} (z_1 z_2)^{-4p} (z_1 - z_2)^{2p} (1 + z_1)^t (1 + z_2)^t v_\lambda = \widetilde{\Phi}_p(t)v_\lambda \\ \text{where } \widetilde{\Phi}_p(t) &= \sum_{i=0}^{2p} (-1)^i \binom{2p}{i} \binom{t}{4p-1-i} \binom{t}{2p+i-1}, \quad t = \langle \lambda, \alpha \rangle. \end{aligned}$$

By using Lemma 3.4 we get that

$$\Phi_p\left(\frac{1}{4p}(t^2 - 2t(p-1))\right) = \widetilde{\Phi}_p(t) = B_p f_p\left(\frac{1}{4p}(t^2 - 2t(p-1))\right),$$

where the non-zero constant

$$B_p = (-1)^p \frac{\binom{2p}{p} (4p)^{3p-1}}{\binom{4p-1}{p} (3p-1)!}.$$

This proves that Φ_p is a non-trivial polynomial of degree $3p - 1$ and that in $A(\overline{M(1)_p})$ we have

$$[Q^2 e^{-2\alpha}] = \Phi_p([\omega]) = B_p f_p([\omega]). \quad (3.14)$$

Since $O(\overline{M(1)_p}) \subset O(\mathcal{W}(p))$, the embedding $\overline{M(1)_p} \subset \mathcal{W}(p)$ induces an algebra homomorphism $A(\overline{M(1)_p}) \rightarrow A(\mathcal{W}(p))$. Applying this homomorphism to relation (3.14) and using Lemma 3.2 we get that $f_p([\omega]) = 0$ in $A(\mathcal{W}(p))$. \square

Corollary 3.6. *The relation*

$$\bar{\omega}^{3p-1} = 0$$

holds inside $\mathcal{W}(p)/C_2(\mathcal{W}(p))$.

Proof. By using Lemma 3.3 and the proof of Theorem 3.5 we obtain

$$Q^2 e^{-2\alpha} = A H_{-2p-1} H + (B_p L(-2)^{3p-1} + w) \cdot \mathbf{1} \quad (3.15)$$

where $A \neq 0$, $B_p \neq 0$ and w is a linear combination of monomials in $\mathcal{U}(\text{Vir}_{\leq -2})$ each containing at least one generator $L(-n)$, $n \geq 3$. Clearly $w \cdot \mathbf{1} \in C_2(\mathcal{W}(p))$. Since

$$H_{-2p-1} H \in C_2(\mathcal{W}(p))$$

and

$$Q^2 e^{-2\alpha} = E_{-2p-1} F + F_{-2p-1} E + 2 H_{-2p-1} H \in C_2(\mathcal{W}(p)),$$

the relation (3.15) implies $\bar{\omega}^{3p-1} = 0$. \square

Remark 3. Notice that Corollary 3.6 gives a better upper bound on $\dim(\mathcal{W}(p)/C_2(\mathcal{W}(p)))$ compared to the one that could be extracted from Theorem 2.1.

The polynomial (3.12) can be rewritten more conveniently as

$$f_p(x) = (x - h_{p,1})(x - h_{2p,1}) \prod_{i=1}^{p-1} (x - h_{i,1})^2 \prod_{i=2p+1}^{3p-1} (x - h_{i,1}). \quad (3.16)$$

Already from (3.16) and $f_p([\omega]) = 0$ it seems feasible to expect some non $L(0)$ -diagonalizable weak $\mathcal{W}(p)$ -modules entering the picture. More precisely, notice that $f_p([\omega])$ will annihilate not only every top component of a $\mathcal{W}(p)$ -module with lowest conformal weight $h_{i,1}$, $1 \leq i \leq 3p-1$, but also any lowest weight subspace of *generalized* conformal weight $h_{i,1}$, $1 \leq i \leq p-1$ with Jordan blocks—with respect to $L(0)$ —of size at most two (cf. second powers in (3.16)). We will get back to this problem in the later sections.

The previous theorem gives also an important information about the possible lowest conformal weights of irreducible $\mathcal{W}(p)$ -modules. Now, we construct a family of $\mathcal{W}(p)$ -modules with lowest conformal weights being precisely the roots of $f_p(x)$.

Recall (1.5). Since $\mathcal{W}(p) \subset V_L$, every (irreducible) V_L -module is of course a $\mathcal{W}(p)$ -module. Thus it is natural to examine the structure of $V_{L+\gamma_i}$, viewed as a $\mathcal{W}(p)$ -module. By using the formula

$$\deg(e^{m\alpha}) = m^2 p - m(p-1),$$

which holds for every $m \in \frac{\mathbb{Z}}{2p}$, we see that

$$\deg(e^{\gamma_i}) = \frac{i(i-2p+2)}{4p} = h_{i+1,1}$$

and

$$\deg(e^{\gamma_{2p-2-i}}) = \frac{i(i-2p+2)}{4p} = h_{i+1,1}.$$

From the last two formulas it is not hard to see that

$$V_{L+\gamma_i}, \quad i \neq 2p-1,$$

have a one-dimensional lowest weight subspace spanned by e^{γ_i} , while

$$V_{L+\gamma_{2p-1}}$$

has a two-dimensional top subspace spanned by $e^{(2p-1)\alpha/2p}$ and $e^{-\alpha/2p}$. We also have an isomorphism

$$V'_{L+\gamma_i} \cong V_{L+\gamma_{2p-2-i}}, \quad (3.17)$$

where $'$ stands for the contragradient dual. Thus the only irreducible self-dual V_L -modules are $V_{L+\gamma_{p-1}}$ and $V_{L+\gamma_{2p-1}}$.

Now, we identify certain submodules and subquotients of each $V_{L+\gamma_{i-1}}$.

For $i = 1, \dots, p-1$ we set

$$\begin{aligned} \Lambda(i) &:= \overline{V_{L+\gamma_{i-1}}} \subset V_{L+\gamma_{i-1}}, \\ \Pi(p-i) &:= \overline{V_{L+\gamma_{2p-i-1}}} \subset V_{L+\gamma_{2p-i-1}}, \end{aligned}$$

and

$$\begin{aligned} \Lambda(p) &= V_{L+\gamma_{p-1}}, \\ \Pi(p) &= V_{L+\gamma_{2p-1}}. \end{aligned}$$

(Here we are actually using the notation introduced in [25].) Thus we obtain the following short exact sequences of vector spaces:

$$\begin{aligned} 0 \longrightarrow \Lambda(i) \longrightarrow V_{L+\gamma_{i-1}} \longrightarrow V_{L+\gamma_{i-1}}/\Lambda(i) \longrightarrow 0, \\ 0 \longrightarrow \Pi(p-i) \longrightarrow V_{L+\gamma_{2p-i-1}} \longrightarrow V_{L+\gamma_{2p-i-1}}/\Pi(p-i) \longrightarrow 0. \end{aligned}$$

It is important to notice that the lowest conformal weight of $\Pi(p-i)$ is

$$h_{2p+i,1} = h_{2p-i,3} = \frac{(i+1)(2p+i-1)}{4p}.$$

Theorem 3.7. *For every $1 \leq i \leq p$, both $\Lambda(i)$ and $\Pi(i)$ are irreducible $\mathcal{W}(p)$ -modules.*

Proof. Assume that $0 \leq i \leq p-1$. In [5] (cf. [38]) we argue that the space of intertwining operators

$$I \left(\begin{matrix} L(c_{p,1}, h) \\ L(c_{p,1}, 2p-1) L(c_{p,1}, h_{i+1,2n+1}) \end{matrix} \right)$$

is non-trivial if and only if $h = h_{i+1,2n-1}$, $h = h_{i+1,2n+1}$ or $h = h_{i+1,2n+3}$, for $n \geq 1$. Since the multiplicities of these fusion rules are always one, we may write formally:

$$\begin{aligned} L(c_{p,1}, 2p-1) \times L(c_{p,1}, h_{i+1,2n+1}) \\ = L(c_{p,1}, h_{i+1,2n-1}) \oplus L(c_{p,1}, h_{i+1,2n+1}) \oplus L(c_{p,1}, h_{i+1,2n+3}) \quad (n \geq 1). \end{aligned} \quad (3.18)$$

The same result has been known by physicists (cf. [19]). The fusion rules (3.18) imply that

$$X_j \Lambda(i+1) \subset \Lambda(i+1), \quad \text{for every } j \in \mathbb{Z}, \text{ where } X = E, F \text{ or } H.$$

Since $\mathcal{W}(p)$ is generated by ω , E , F and H we have that $\Lambda(i+1)$ is a $\mathcal{W}(p)$ -module. In order to prove that $\Lambda(i+1)$ is irreducible, we shall first prove that e^{γ_i} is a cyclic vector in $\Lambda(i+1)$, i.e., $\Lambda(i+1) = \langle e^{\gamma_i} \rangle$. Assume that $\langle e^{\gamma_i} \rangle \neq \Lambda(i+1)$. This implies that $M = \Lambda(i+1)/\langle e^{\gamma_i} \rangle$ is a non-trivial $\mathbb{Z}_{\geq 0}$ -graded $\mathcal{W}(p)$ -module. By Theorem 3.5 it follows that $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(h' + n)$, where $h' = h_{j+1,1}$ for $0 \leq j \leq 3p-2$. Recall that, as a module for the Virasoro algebra, $\Lambda(i+1)$ is generated by the family of singular vectors of weights $h_{i+1,2n+1}$, $n \in \mathbb{Z}_{\geq 0}$. Therefore every non-trivial vector from the top level $\bar{v} \in M(h')$ has the form $\bar{v} = v + \langle e^{\gamma_i} \rangle$, where v is a singular vector for the Virasoro algebra of weight h' . This leads to a contradiction since $h' = h_{i+1,2n+1}$ if and only if $n = 0$, and every vector of weight $h_{i+1,1}$ in $\Lambda(i+1)$ is proportional to $e^{\gamma_i} \in \langle e^{\gamma_i} \rangle$. Thus $\Lambda(i+1) = \langle e^{\gamma_i} \rangle$.

Assume that N is any non-trivial $\mathcal{W}(p)$ -submodule of $\Lambda(i+1)$. Then N is also a $\mathbb{Z}_{\geq 0}$ -graded $\mathcal{W}(p)$ -module. Again, by Theorem 3.5, $N = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} N(h' + n)$, where $h' = h_{j+1,1}$ for $0 \leq j \leq 3p-2$. The same arguments as before show that $h' = h_{i+1,1}$, which implies that $e^{\gamma_i} \in N$ and therefore $N = \Lambda(i)$.

By using the fusion rules, one can also prove that $\Pi(p-i)$ is a $\mathcal{W}(p)$ -module, where $0 \leq i \leq p-1$. The top component $\Pi(p-i)(0)$ is an irreducible two-dimensional $A(\mathcal{W}(p))$ -module with conformal weight $h_{2p+i,1} = h_{2p-i,3}$. Theorem 1.2 implies that as a module for the Virasoro algebra $\Pi(p-i)$ is generated by the family of singular vectors of weights $h_{2p-i,2n+1}$, $n \geq 1$.

Let $j \in \{1, \dots, 3p-1\}$. By using the fact that

$$h_{2p-i,2n+1} = h_{j,1} \quad \text{iff} \quad n = 1, \quad j = 2p + i,$$

and a completely analogous proof as in the case of $\Lambda(i+1)$ one can prove that $\Pi(p-i)$ is an irreducible $\mathcal{W}(p)$ -module. \square

Applying the previous theorem in the case of $\mathcal{W}(p) = \Lambda(1)$ we get:

Corollary 3.8. *The vertex operator algebra $\mathcal{W}(p)$ is simple.*

The next result will be proven in Theorem 5.1.

Proposition 3.9. *In Zhu's associative algebra we have*

$$[H] * [F] - [F] * [H] = -2q([\omega])[F], \quad (3.19)$$

$$[H] * [E] - [E] * [H] = 2q([\omega])[E], \quad (3.20)$$

$$[E] * [F] - [F] * [E] = -2q([\omega])[H] \quad (3.21)$$

where q is a polynomial of degree at most $p - 1$.

It is known that the top component $M(0)$ of a $\mathbb{Z}_{\geq 0}$ -gradable $\mathcal{W}(p)$ -module M carries a structure of $A(\mathcal{W}(p))$ -module. In particular, the top components $\Pi(i)(0)$ and $\Lambda(i)(0)$ are $A(\mathcal{W}(p))$ -modules. More precisely, the top component of $\Lambda(i)$ is 1-dimensional and has conformal weight $h_{i,1}$ for $i = 1, \dots, p$. On the other hand, the top component of $\Pi(i)$ is 2-dimensional and has conformal weight $h_{p+i,3} = h_{3p-1-i,1}$ for $i = 1, \dots, p$. So we get:

Proposition 3.10.

- (1) For every $1 \leq i \leq p$ the top component $\Lambda(i)(0)$ of $\Lambda(i)$ has lowest conformal weight $h_{i,1}$. Moreover $\Lambda(i)(0)$ is an irreducible 1-dimensional $A(\mathcal{W}(p))$ -module spanned by the vector $e^{\gamma_{i-1}}$.
- (2) For every $2p \leq i \leq 3p - 1$ the top component $\Pi(3p - i)(0)$ of $\Pi(3p - i)$ has lowest conformal weight $h_{i,1}$. Moreover, $\Pi(3p - i)(0)$ is an irreducible 2-dimensional $A(\mathcal{W}(p))$ -module spanned by the vectors $e^{\gamma_{2p-1-i}}$ and $Qe^{\gamma_{2p-1-i}}$.

Theorem 3.11. *The set*

$$\{\Pi(i)(0): 1 \leq i \leq p\} \cup \{\Lambda(p - i)(0): 1 \leq i \leq p\}$$

provides, up to isomorphism, all irreducible modules for Zhu's algebra $A(\mathcal{W}(p))$.

Proof. Assume that U is an irreducible $A(\mathcal{W}(p))$ -module. Relation $f_p([\omega]) = 0$ in $A(\mathcal{W}(p))$ implies that

$$L(0)|U = h_{i,1}\text{Id}, \quad \text{for } i \in \{1, \dots, p\} \cup \{2p, \dots, 3p - 1\}.$$

Assume first that $2p \leq i \leq 3p - 1$. By combining Propositions 3.9 and 3.10 we have that $q(h_{i,1}) \neq 0$. Define

$$e = \frac{1}{\sqrt{2}q(h_{i,1})}E, \quad f = -\frac{1}{\sqrt{2}q(h_{i,1})}F, \quad h = \frac{1}{q(h_{i,1})}H.$$

Therefore U carries the structure of an irreducible, \mathfrak{sl}_2 -module with the property that $e^2 = f^2 = 0$ and $h \neq 0$ on U . This easily implies that U is 2-dimensional irreducible \mathfrak{sl}_2 -module. Moreover, as an $A(\mathcal{W}(p))$ -module U is isomorphic to $\Pi(3p - i)(0)$.

Assume next that $1 \leq i \leq p$. If $q(h_{i,1}) \neq 0$, as above we conclude that U is an irreducible 1-dimensional \mathfrak{sl}_2 -module. Therefore $U \cong \Lambda(i)(0)$.

If $q(h_{i,1}) = 0$ from Proposition 3.9 we have that the action of generators of $A(\mathcal{W}(p))$ commute on U . Irreducibility of U implies that U is 1-dimensional. Since $[H], [E]^2, [F]^2$ must act trivially on U , we conclude that $[H], [E], [F]$ also act trivially on U . Therefore $U \cong \Lambda(i)(0)$. \square

Remark 4. In Theorem 5.1 below we shall see that in fact $q(h_{i,1}) \neq 0$ for every $i \in \{1, \dots, p\}$, which can be used to give a shorter proof of Theorem 3.11.

Applying Zhu's theory [42] on Theorem 3.11 and using irreducibility result from Theorem 3.7 we get the classification of all irreducible $\mathcal{W}(p)$ -modules (the same result was stated in [25]).

Theorem 3.12. *The set*

$$\{\Pi(i): 1 \leq i \leq p\} \cup \{\Lambda(p-i): 1 \leq i \leq p\}$$

provides, up to isomorphism, all irreducible modules for the vertex operator algebra $\mathcal{W}(p)$.

4. A description of the category of ordinary $\mathcal{W}(p)$ -modules

In the previous section we classified simple objects in the category of $\mathcal{W}(p)$ -modules. Here we derive additional results about reducible modules. In [5] we classified irreducible self-dual $\overline{M(1)_p}$ -modules. For the triplet vertex algebra we have a very different result.

Corollary 4.1. *All irreducible $\mathcal{W}(p)$ -modules are self-dual.*

Proof. It is sufficient to show that $\Pi(i)'(0) \cong \Pi(i)(0)$ and $\Lambda(i)'(0) \cong \Lambda(i)(0)$, where W' denotes the dual module of W . Both isomorphisms follow directly from Theorem 3.7 and $h_{i,1} \neq h_{j,1}$, $i \neq j$, $i, j \in \{1 \leq k \leq p\} \cup \{2p \leq k \leq 3p-1\}$. \square

Proposition 4.2. *For $1 \leq i \leq p-1$ we have the following non-split short exact sequences of $\mathcal{W}(p)$ -modules:*

$$0 \longrightarrow \Lambda(i) \longrightarrow V_{L+(i-1)\alpha/2p} \longrightarrow \Pi(p-i) \longrightarrow 0, \quad (4.22)$$

$$0 \longrightarrow \Pi(p-i) \longrightarrow V_{L+(2p-i-1)\alpha/2p} \longrightarrow \Lambda(i) \longrightarrow 0. \quad (4.23)$$

Proof. From Theorem 3.7 we know that both $\Lambda(i)$ and $\Pi(i)$ are irreducible $\mathcal{W}(p)$ -modules. Consider $\Psi(p-i) := V_{L+\gamma_{i-1}}/\Lambda(i)$. It is clear (cf. Theorems 1.1 and 1.2) that as a Virasoro algebra module $\Psi(p-i)$ is isomorphic to $\Pi(p-i)$. Also, as $A(\mathcal{W}(p))$ -modules $\Psi(p-i)(0) \cong \Pi(p-i)(0)$. Now, irreducibility of $\Pi(p-i)$ gives $\Psi(p-i) \cong \Pi(p-i)$. Thus, we have (4.22). Similarly, we show (4.23). From Theorems 1.1 and 1.2 it is clear that the sequences are non-split. \square

The previous proposition seems to be known as well as the next result (cf. [25]).

Now, we recall several standard facts about category \mathcal{O}_c for the Virasoro algebra. A Virasoro module M is said to be an object in \mathcal{O}_c if

- (i) $M = \bigoplus_{n \in \mathbb{C}} M_n$, $M_n = \{v \in M: L(0)v = nv\}$, $\dim(M_n) < +\infty$.
- (ii) The central element acts as multiplication by c .
- (iii) There exist $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ such that if $M_n \neq 0$ then $n \geq \lambda_i$ for some i .

We denote by $M(c, h)$ the Verma Virasoro module with lowest conformal weight h and central charge c . We introduce a partial ordering " \preceq " on the set of weights ($= \mathbb{C}$). We say that $h' \preceq h$ if

$$L(c, h') \text{ is a subquotient of } M(c, h).$$

Extend the partial ordering “ \preceq ” to an equivalence relation \sim on the set of weights. Let $[h]$ denote an equivalence class and \mathcal{F} the set of all equivalence classes. Then it is known that every $W \in \text{Obj}(\mathcal{O}_c)$ admits *block decomposition*:

$$W = \bigoplus_{[h] \in \mathcal{F}} W_{[h]}, \quad W_{[h]} \in \mathcal{O}_c^{[h]},$$

where $M \in \mathcal{O}_c^{[h]}$ if every irreducible subquotient $L(c, h')$ of M satisfies $h' \in [h]$. Each $\mathcal{O}_c^{[h]}$ is in fact a full subcategory of \mathcal{O}_c .

We would like to obtain a similar description for the category of *ordinary* (i.e., non-logarithmic) $\mathcal{W}(p)$ -modules. Since $\mathcal{W}(p)$ is not a Lie algebra with triangular decomposition one cannot define $\mathcal{W}(p)$ -blocks as above by using Verma modules.

Lemma 4.3. *Every ordinary $\mathcal{W}(p)$ -module, viewed as a Virasoro algebra module, is an object in $\mathcal{O}_{c_{p,1}}$.*

Proof. We only have to check that there exist $\lambda_1, \dots, \lambda_k$ but this is clear since $\mathcal{W}(p)$ is C_2 -cofinite. \square

The previous lemma indicates that we should first try to describe Virasoro blocks for irreducible $\mathcal{W}(p)$ -modules.

The first important observation we make is that each irreducible $\mathcal{W}(p)$ -module belongs to a unique (Virasoro) block. This is a consequence of Theorems 1.1 and 1.2, and the fact that

$$h_{i,1} \notin [h_{j,1}], \quad \text{for } i, j \in \{1, \dots, p\} \cup \{2p\}, \quad i \neq j. \quad (4.24)$$

Actually we can say more: $\Lambda(p)$ and $\Pi(p)$ live in two distinct blocks and the remaining irreducible modules are distributed in additional $p - 1$ blocks, such that $\Lambda(i)$ and $\Pi(p - i)$ are in the same block. These $p + 1$ blocks are represented by $h_{i,1}$, $i = 1, \dots, p - 1$, $h_{p,1}$ and $h_{2p,1}$. Our next result says that the same decomposition persists at the level of $\mathcal{W}(p)$ -modules.

Theorem 4.4. *The category of ordinary $\mathcal{W}(p)$ -modules contains precisely $p + 1$ blocks, so that every $\mathcal{W}(p)$ -module W decomposes as a direct sum of $\mathcal{W}(p)$ -modules:*

$$W = W_{[h_{1,1}]} \oplus \dots \oplus W_{[h_{p,1}]} \oplus W_{[h_{2p,1}]}, \quad (4.25)$$

where

$$W_{[h_{i,1}]} \in \text{Obj}(\mathcal{O}_{c_{p,1}}^{[h_{i,1}]}).$$

Proof. We certainly have a decomposition of W on the level of Virasoro modules (with possibly additional summands). In view of the previous discussion it suffices to show that F , H and E preserve the summands $W_{[h_{i,1}]}$ and that there are no additional blocks appearing in (4.25). Since E , F and H are primary fields of lowest conformal weight $2p - 1$ the action of $\mathcal{W}(p)$ on W defines a $L(c_{p,1}, 0)$ -intertwining operator of type

$$\begin{pmatrix} W \\ L(c_{p,1}, 2p - 1)W_{[h_{i,1}]} \end{pmatrix}.$$

If W admits a non-trivial summand from a different block it would follow that there is a non-trivial intertwining operator of the type

$$\begin{pmatrix} W_{[h_{j,1}]} \\ L(c_{p,1}, 2p-1)W_{[h_{i,1}]} \end{pmatrix}, \quad h_{j,1} \neq h_{i,1}.$$

But this will contradict to the fusion rules formula (3.18). It follows that E , F and H preserve each block. To show that there are no additional blocks one uses the classification of irreducible $\mathcal{W}(p)$ -modules from Theorem 3.12. \square

Since each Virasoro block contains at least one irreducible $\mathcal{W}(p)$ -module and there is a non-trivial extension of $\Lambda(i)$ and $\Pi(p-i)$ inside the single block, (4.25) is the proper block decomposition of a $\mathcal{W}(p)$ -module. Correspondingly, the category of $\mathcal{W}(p)$ -modules is a block preserving, subcategory of $\mathcal{O}_{c_{p,1}}$.

Corollary 4.5. *For every $p \geq 2$ and $i \neq j$ we have*

$$\begin{aligned} \text{Ext}_{\mathcal{W}(p)}^1(\Lambda(i), \Lambda(j)) &= \text{Ext}_{\mathcal{W}(p)}^1(\Pi(i), \Pi(j)) = 0, \\ \text{Ext}_{\mathcal{W}(p)}^1(\Lambda(i), \Pi(p-j)) &= \text{Ext}_{\mathcal{W}(p)}^1(\Pi(i), \Lambda(p-j)) = 0, \end{aligned}$$

where the Ext-groups are computed inside the category of ordinary $\mathcal{W}(p)$ -modules.

This result has been stated in [25].

5. The structure of $A(\mathcal{W}(p))$

In this section we derive additional relations in Zhu's algebra needed for a better description of $A(\mathcal{W}(p))$. Some relations obtained here could be also used for classification of irreducible modules in a simpler fashion.

The following important theorem gives a fairly explicit description of $A(\mathcal{W}(p))$ in terms of generators and relations.

Theorem 5.1. *The Zhu's associative algebra is generated by $[\omega]$, $[F]$, $[H]$, and $[E]$. Also, we have the following relations in $A(\mathcal{W}(p))$:*

- (i) $[E]^2 = [F]^2 = 0$.
- (ii) $[H]^2 = C_p P([\omega])$, where

$$P(x) = (x - h_{p,1}) \prod_{i=1}^{p-1} (x - h_{i,1})^2 \in \mathbb{C}[x]$$

and C_p is a non-zero rational number.

$$(iii) \quad \begin{aligned} [H] * [F] &= -[F] * [H] = -q([\omega]) * [F], \\ [H] * [E] &= -[E] * [H] = q([\omega]) * [E], \end{aligned}$$

where $q(x)$ is a non-zero polynomial of degree $\leq p-1$ and

$$q(h_{i,1}) \neq 0, \quad 1 \leq i \leq p.$$

(iv)

$$\begin{aligned} [H] * [F] - [F] * [H] &= -2q([\omega])[F], \\ [H] * [E] - [E] * [H] &= 2q([\omega])[E], \\ [E] * [F] - [F] * [E] &= -2q([\omega])[H], \end{aligned}$$

where $q(x)$ is as in (iii).

$$(v) \quad \prod_{i=2p}^{3p-1} ([\omega] - h_{i,1}) * [X] = 0, \quad X \in \{E, F, H\}.$$

Proof. By Proposition 3.1 it is clear that $A(\mathcal{W}(p))$ is generated by $[\omega]$, $[F]$, $[H]$ and $[E]$.

Part (i) is trivial to show. Part (ii) has been established earlier in [4]. It remains to show (iii), (iv) and (v).

Observe that the screening Q preserves $O(\mathcal{W}(p))$, hence it acts as a derivation of $A(\mathcal{W}(p))$. We will abuse the notation and use the same letter for the projection of Q on $A(\mathcal{W}(p))$, so in particular

$$Q[F] = [H], \quad Q[H] = [E].$$

Then

$$0 = Q([X] * [X]) = [H] * [X] + [X] * [H], \quad X \in \{E, F\},$$

which yields two (easy) assertions in (iii). If we can show that

$$[H] * [F] = -q([\omega]) * [F], \tag{5.26}$$

then the formulas

$$Q^2([H] * [F]) = [H] * [E] + 2[E] * [H] = -[H] * [E]$$

$$Q^2(q([\omega] * [F])) = q([\omega]) * [E]$$

yield

$$[H] * [E] = q([\omega]) * [E].$$

The proof of relations (5.26) with $q(x)$ satisfying the properties as in the theorem is rather technical and it is given in Appendix A at the end of the paper.

For (iv), the first two relations follow directly from (iii), and for the third observe that

$$Q([H] * [F] - [F] * [H]) = [E] * [F] - [F] * [E] = -2q([\omega]) * [H].$$

The proof follows.

Next we prove part (v). By part (iii), $q(h_{i,1}) \neq 0$, for $i = 1, \dots, p$. Hence $q([\omega])$ is a unit in $A(\mathcal{W}(p))$. Since the Q -screening preserves $O(\mathcal{W}(p)) \subset \mathcal{W}(p)$, and commutes with the action of the Virasoro algebra, the relation

$$l([\omega]) * [F] = 0, \quad l([\omega]) = \prod_{i=2p}^{3p-1} ([\omega] - h_{i,1})$$

implies

$$l([\omega]) * [X] = 0, \quad X \in \{E, F, H\}.$$

Suppose on the contrary that

$$l([\omega]) * [F] \neq 0.$$

Since

$$0 = Q^2([F]^2) = [E] * [F] + [F] * [E] + 2[H]^2,$$

by using (ii) we get

$$[E] * [F] + [F] * [E] = -2C_p P([\omega]). \quad (5.27)$$

The last assertion gives

$$[F] * [E] = q([\omega]) * [H] - C_p P([\omega]). \quad (5.28)$$

Thus, we have

$$l([\omega]) * [F] * [E] = l([\omega]) (q([\omega]) * [H] - C_p P([\omega])) = l([\omega]) * q([\omega]) * [H], \quad (5.29)$$

where in the last equality we used

$$l([\omega]) * C_p P([\omega]) = C_p f_p([\omega]) = 0,$$

which holds in $A(\mathcal{W}(p))$. After we multiply (5.29) with $[F]$, from the left and use $[F]^2 = 0$, and Theorem 5.1(iii), we obtain

$$0 = l([\omega]) * q([\omega]) * [F] * [H] = l([\omega]) * q([\omega])^2 * [F].$$

But since $q([\omega])^2$ is a unit in $A(\mathcal{W}(p))$ we obtain

$$l([\omega]) * [F] = 0,$$

a contradiction. \square

Corollary 5.2. *Zhu's algebra $A(\mathcal{W}(p))$ contains a Lie subalgebra isomorphic to \mathfrak{sl}_2 . Correspondingly, any $A(\mathcal{W}(p))$ -module is naturally an \mathfrak{sl}_2 -module.*

Proof. Since $q(h_{i+1,1}) \neq 0$ for every i , $0 \leq i \leq 3p-2$, we have that $q(x)$ is relatively prime with $f_p(x)$ and therefore $q([\omega])$ is a unit in $A(\mathcal{W}(p))$. Define non-zero vectors

$$e = \frac{1}{\sqrt{2}}q([\omega])^{-1}E, \quad f = -\frac{1}{\sqrt{2}}q([\omega])^{-1}F, \quad h = q([\omega])^{-1}H.$$

It is easy to see that $[e, f] = h$, $[h, f] = -2f$ and $[h, e] = 2e$ holds. Thus $\text{span}\{e, f, h\}$ is isomorphic to \mathfrak{sl}_2 . This also implies that any $A(\mathcal{W}(p))$ -module, in particular $A(\mathcal{W}(p))$ itself, is an \mathfrak{sl}_2 -module. \square

Corollary 5.3. *The associative algebra $A(\mathcal{W}(p))$ is spanned by*

$$\{[\omega]^i, 0 \leq i \leq 3p-2\} \cup \{[\omega]^i * [X], 0 \leq i \leq p-1, X = E, F \text{ or } H\}.$$

Thus, $A(\mathcal{W}(p))$ is at most $(6p-1)$ -dimensional.

Our goal is to describe $A(\mathcal{W}(p))$ as a sum of ideals. For these purposes, for $i = 1, \dots, p-1$, let

$$v_i = (\lambda_i[\omega] + v_i) \prod_{j=1, \dots, 3p-1; j \neq i, j \neq 2p-i} ([\omega] - h_{j,1}),$$

where λ_i and v_i are unspecified constants and

$$w_i = \prod_{j=1, \dots, 3p-1; j \neq i} ([\omega] - h_{j,1}),$$

$$v_i * w_i = w_i * v_i = w_i.$$

We clearly have

$$v_i * v_j = 0, \quad i \neq j,$$

and

$$w_i^2 = 0, \quad i = 1, \dots, p-1.$$

Lemma 5.4. For every $i = 1, \dots, p-1$ there are constants λ_i, v_i such that

$$v_i = (\lambda_i [\omega] + v_i) \prod_{j=1, \dots, 3p-1; j \neq i, j \neq 2p-i} ([\omega] - h_{j,1}) \neq 0$$

satisfy

$$v_i * v_j = \delta_{i,j} v_i,$$

and

$$w_i * v_j = v_j * w_i = \delta_{i,j} w_i,$$

here $\delta_{i,j}$ is the Kronecker symbol.

Proof. We let

$$\tilde{v}_i = \prod_{j=1, \dots, 3p-1; j \neq i, j \neq 2p-i} ([\omega] - h_{j,1}),$$

which is certainly non-zero (since it has a non-trivial action on an irreducible $A(\mathcal{W}(p))$ -module). Fix an index i , $1 \leq i \leq p-1$. We have to show that λ_i and v_i exist and that $\lambda_i h_i + v_i \neq 0$, which is sufficient to argue that $v_i \neq 0$. From the very definition it is easy to see that for every $f(x) \in \mathbb{C}[x]$ we have

$$f([\omega]) * w_i = f(h_i) w_i$$

and clearly

$$([\omega] - h_{i,1}) * \tilde{v}_i = w_i.$$

The last two formulas yield

$$\begin{aligned} f([\omega]) * \tilde{v}_i &= (f(h_i) + ([\omega] - h_{i,1})r([\omega])) * \tilde{v}_i \\ &= f(h_{i,1}) * \tilde{v}_i + r(h_{i,1})w_i, \end{aligned}$$

where $r(x)$ is the unique polynomial satisfying

$$f(x) = r(x)(x - h_{i,1}) + f(h_{i,1}).$$

Specialize now

$$f(x) = \prod_{j=1, \dots, 3p-1; j \neq i, j \neq 2p-i} (x - h_{j,1}),$$

so that

$$\begin{aligned}\tilde{v}_i * \tilde{v}_i &= f(h_{i,1})\tilde{v}_i + r(h_{i,1})w_i, \\ \tilde{v}_i * w_i &= f(h_{i,1})w_i,\end{aligned}\tag{5.30}$$

and consequently

$$\begin{aligned}v_i * v_i &= (\lambda_i[\omega] + v_i)^2(\tilde{v}_i * \tilde{v}_i) \\ &= f(h_{i,1})((\lambda_i h_{i,1} + v_i)^2)\tilde{v}_i + 2\lambda_i f(h_{i,1})(\lambda_i h_{i,1} + v_i)w_i + r(h_{i,1})(\lambda_i h_{i,1} + v_i)^2 w_i \\ &= f(h_{i,1})((\lambda_i h_{i,1} + v_i)^2)\tilde{v}_i + (\lambda_i h_{i,1} + v_i)(2\lambda_i f(h_{i,1}) + r(h_{i,1})(\lambda_i h_{i,1} + v_i))w_i \\ &= (K + L([\omega] - h_{i,1})) * \tilde{v}_i\end{aligned}$$

where we let

$$K = f(h_{i,1})((\lambda_i h_{i,1} + v_i)^2)$$

and

$$L = (\lambda_i h_{i,1} + v_i)(2\lambda_i f(h_{i,1}) + r(h_{i,1})(\lambda_i h_{i,1} + v_i)).$$

Since we want

$$v_i * v_i = v_i,$$

by comparing the coefficients we get a system

$$L = \lambda_i, \quad K - Lh_{i,1} = v_i.$$

From the second equation we obtain

$$\lambda_i h_{i,1} + v_i = \frac{1}{f(h_{i,1})} \neq 0.\tag{5.31}$$

From this relation, after some computation, we get

$$\lambda_i = -\frac{r(h_{i,1})}{f(h_{i,1})^2},$$

and

$$v_i = \frac{1}{f(h_{i,1})} + \frac{h_{i,1}r(h_{i,1})}{f(h_{i,1})^2}.$$

From (5.30) and (5.31) we clearly have

$$v_i * w_i = w_i * v_i = (\lambda_i h_{i,1} + v_i)(\tilde{v}_i * w_i) = w_i. \quad \square$$

Example 5.5. For instance, for $p = 2$ and $i = 1$, we have $h_{1,1} = 0$ and

$$\tilde{v}_1 = ([\omega] + 1/8)([\omega] - 3/8)([\omega] - 1).$$

From the previous lemma we have $r(0) = 13/64$, $f(0) = 3/64$

$$\lambda_1 = -\frac{832}{9}, \quad v_i = \frac{64}{3},$$

and hence

$$v_1 = -\frac{64}{9}(13[\omega] - 3)([\omega] + 1/8)([\omega] - 3/8)([\omega] - 1).$$

Remark 5. It is not at all clear that $w_i \neq 0$!

For $2p \leq i \leq 3p - 1$, let $\mathbb{M}_{h_{i,1}}$ be the vector space spanned by the vectors

$$\begin{aligned} A^{(i)} &= C_p \prod_{j=2p, \dots, 3p-1, j \neq i} ([\omega] - h_{j,1}) * P([\omega]), \\ B^{(i)} &= \prod_{j=2p, \dots, 3p-1, j \neq i} ([\omega] - h_{j,1}) * [H], \\ C^{(i)} &= \prod_{j=2p, \dots, 3p-1, j \neq i} ([\omega] - h_{j,1}) * [E], \\ D^{(i)} &= \prod_{j=2p, \dots, 3p-1, j \neq i} ([\omega] - h_{j,1}) * [F]. \end{aligned}$$

It is easy to see that $\mathbb{M}_{h_{i,1}}$ is a non-trivial vector space. From Theorem 5.1 it follows that $[\omega]$ acts on $\mathbb{M}_{h_{i,1}}$ by the scalar $h_{i,1}$.

Lemma 5.6.

- (i) For every $i \in \mathbb{Z}_{\geq 0}$, $2p \leq i \leq 3p - 1$, $\mathbb{M}_{h_{i,1}}$ is an ideal in $A(\mathcal{W}(p))$ isomorphic to the matrix algebra $M_2(\mathbb{C})$.
- (ii) $\bigoplus_{i=2p}^{3p-1} \mathbb{M}_{h_{i,1}}$ is an ideal in $A(\mathcal{W}(p))$.

Proof. Theorem 5.1 implies that $\mathbb{M}_{h_{i,1}}$ is an ideal in $A(\mathcal{W}(p))$. Moreover, irreducible $A(\mathcal{W}(p))$ -module $\Pi(3p - i)(0)$ is also an irreducible 2-dimensional $\mathbb{M}_{h_{i,1}}$ -module. Thus we have a homomorphism from $\mathbb{M}_{h_{i,1}}$ to $M_2(\mathbb{C})$. But this has to be an isomorphism, because the module is irreducible. This proves (i). Assertion (ii) follows from (i) and the fact that $[\omega]$ acts on $\mathbb{M}_{h_{i,1}}$ by the scalar $h_{i,1}$. \square

Define:

$$v_p = \prod_{j=1, \dots, 3p-1, j \neq p} ([\omega] - h_{j,1}); \quad \mathbb{C}_{h_{p,1}} = \mathbb{C}v_p.$$

We have:

$$v_p * v_i = v_p * w_i = 0, \quad i = 1, \dots, p-1,$$

and

$$v_p * [\omega] = h_{p,1} v_p.$$

Also, it is easy to see (by using Theorem 5.1(v)) that

$$X * v_p = v_p * X = 0, \quad X \in \{A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}\}.$$

Thus, we have

Lemma 5.7. $\mathbb{C}_{h_{p,1}}$ is a one-dimensional ideal in $A(\mathcal{W}(p))$.

Let $\mathbb{I}_{h_{i,1}}$ be the ideal in $A(\mathcal{W}(p))$ spanned by v_i and w_i . We expect the following to be true.

Conjecture 5.8. Each $\mathbb{I}_{h_{i,1}}$ is a two-dimensional ideal.

We will return to Conjecture 5.8 in the next section.

Now, we summarize the results from this section.

Theorem 5.9. Zhu's algebra $A(\mathcal{W}(p))$ decomposes as a direct sum of ideals

$$A(\mathcal{W}(p)) = \bigoplus_{i=2p}^{3p-1} \mathbb{M}_{h_{i,1}} \oplus \bigoplus_{i=1}^{p-1} \mathbb{I}_{h_{i,1}} \oplus \mathbb{C}_{h_{p,1}},$$

where $\mathbb{M}_{h_{i,1}}$, $\mathbb{I}_{h_{i,1}}$ and $\mathbb{C}_{h_{p,1}}$ are as above. Assume that Conjecture 5.8 holds. Then

$$\dim(A(\mathcal{W}(p))) = 6p - 1.$$

6. Logarithmic $\mathcal{W}(p)$ -modules

In this section we prove the existence of certain logarithmic $\mathcal{W}(p)$ -modules needed for the description of ideals $\mathbb{I}_{h_{i,1}}$.

Let us recall that a logarithmic module for a vertex operator algebra is a weak $\mathcal{W}(p)$ -module which admits a decomposition into generalized $L(0)$ -eigenspaces [39]. Since $\mathcal{W}(p)$ satisfies the C_2 -property a result of Miyamoto [41] implies that every weak $\mathcal{W}(p)$ -module is logarithmic. Non-trivial logarithmic modules (in the sense that they admit non-trivial Jordan blocks) are always reducible. It is a priori not clear that non-trivial logarithmic $\mathcal{W}(p)$ -modules actually exist, so it is still an open problem to construct them explicitly (we feel that the approach from [18], [39] and [40] might be useful for those purposes). For $p = 2$ a single logarithmic module can be constructed explicitly by using symplectic fermions as shown in [27] (cf. also [1].) An additional difficulty with logarithmic modules is that they might involve Jordan blocks of larger size deeper in the grading compared with those on the top. Thus, the lowest weight subspace, being a $A(\mathcal{W}(p))$ -module, does not carry enough information about the module itself and it cannot be

used to give even an upper bound on the size of Jordan blocks. Luckily there are higher analogs of Zhu's algebra, denoted by $A_n(V)$, $n \geq 1$, that control the representation theory below the top component. We plan to return to $A_n(\mathcal{W}(p))$ in forthcoming publications [6].

Even though in this paper we did not develop proper algebraic tools to study logarithmic modules, we do have analytic tools stemming from generalized graded traces of weak $\mathcal{W}(p)$ -modules [41]. So in what follows we shall try to prove the existence of logarithmic modules by using an indirect approach. Our proof is in the spirit of the proof of existence of g -twisted sectors for g -rational vertex operator algebras obtained by Dong, Li and Mason in [12]. In their approach, non-triviality of g -twisted sectors (i.e., existence of a g -twisted module) is proven by using modular invariance. Similarly, here we employ Miyamoto's modular invariance of *pseudotraces* for vertex algebras satisfying C_2 -cofiniteness condition. We should say here that Miyamoto's result provides us only with *some* logarithmic modules. The hard part is to construct *enough* logarithmic $\mathcal{W}(p)$ -modules with two-dimensional lowest weight subspaces, such that Conjecture 5.8 holds true. We will eventually show that this could be done, without too much effort, for p being a prime integer.

We first recall some notation and results from [41]. For every $n \geq 0$ the n th Zhu's associative algebra is defined as $A_n(V) = V/O_n(V)$, where $O_n(V)$ is spanned by

$$\text{Res}_x \frac{(1+x)^{n+\deg(u)}}{x^{2n+2}} Y(u, x)v,$$

where $u, v \in V$ (with u homogeneous), and the n th product $*_n$ in $A_n(V)$ is defined similarly as for $A_0(V) = A(V)$ (see [41] for details). If V is C_2 -cofinite then all $A_n(V)$ are in fact finite-dimensional associative algebras. If $W = \coprod_{n \geq 0} W_n$ is a $\mathbb{Z}_{\geq 0}$ -graded weak V -module then $W([0, n]) = \bigoplus_{0 \leq i \leq n} W_i$ is an $A_n(V)$ -module and $o([a] *_n [b]) = o(a)o(b)$ holds on $W([0, n])$. Suppose that T is an $A_n(V)$ -module. Then one forms a generalized Verma V -module $W_T(n)$ (a weak V -module), whose n th graded piece $W(n)$ is T . Another important gadget is a *pseudo-trace* $\text{tr}_{W_T(n)}^\phi$, where ϕ is a symmetric map interlocked with $W_T(n)$ (for definitions see [41]). The main result of Miyamoto is then

Theorem 6.1. *Suppose that V is C_2 -cofinite VOA. Let n be large enough and W^1, \dots, W^m be n th generalized Verma V -module, interlocked with symmetric functions ϕ_i . Then the vector space spanned by $\text{tr}_{W^i}^{\phi_i} q^{L(0)-c/24}$, is modular invariant. In addition, in the τ expansion of $\text{tr}_{W^i}^{\phi_i} q^{L(0)-c/24}$ all coefficients are ordinary (pseudo)characters.*

Since $L(0)$ does not act semisimply in general in all these formulas with pseudotraces we use

$$q^{L(0)} = \sum_{k \geq 0} (2\pi i \tau)^k \frac{L_n^k(0)}{k!} q^{L_{ss}(0)},$$

$L(0) = L_{ss}(0) + L_n(0)$, where $L_n(0)$ is the nilpotent and $L_{ss}(0)$ the semisimple part of $L(0)$.

Now, we take V to be $\mathcal{W}(p)$ and discuss first the irreducible $\mathcal{W}(p)$ -characters. We denote by

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n),$$

the Dedekind η -function and by

$$\theta_{i,p}(q) = \sum_{n \in \mathbb{Z}} q^{\frac{(2pn+i)^2}{4p}}, \quad (6.32)$$

$$(\partial\theta)_{i,p}(q) = \sum_{n \in \mathbb{Z}} (2pn+i) q^{\frac{(2pn+i)^2}{4p}}, \quad (6.33)$$

certain theta constants and their derivatives (these are in fact modular forms of weight $1/2$ and $3/2$ for some congruence subgroups, respectively). From Theorems 1.1 and 1.2 it is not hard to see (by using well-known formulas for the characters of irreducible Virasoro modules) that in fact (cf. [19,21,25])

$$\mathrm{tr}_{\Lambda(i)} q^{L(0)-c_{p,1}/24} = \frac{1}{p\eta(q)} (i\theta_{p,p-i}(q) + (\partial\theta_{p,p-i})(q)), \quad (6.34)$$

$$\mathrm{tr}_{\Pi(i)} q^{L(0)-c_{p,1}/24} = \frac{1}{p\eta(q)} (i\theta_{p,i}(q) - (\partial\theta_{p,i})(q)). \quad (6.35)$$

In what follows, we will say that two q -series $f \in q^a \mathbb{C}[[q]]$ and $g \in q^b \mathbb{C}[[q]]$ have *compatible* q -expansions if $a - b \in \mathbb{Z}$.

Lemma 6.2. *Let $p \geq 2$ be a prime integer. Then for $i \neq j$, $i, j \in \{1 \leq i \leq p\} \cup \{2p \leq i \leq 3p-1\}$,*

$$h_{i,1} - h_{j,1} \in \mathbb{Z}$$

if and only if $i - j = 2p$, $1 \leq i \leq p-1$ (or $i - j = -2p$).

Proof. The statement clearly holds for $p = 2$, so we may assume $p \geq 3$. Since p is prime

$$h_{i,1} - h_{j,1} = \frac{(i-p)^2 - (j-p)^2}{4p} \in \mathbb{Z},$$

implies that p divides $i + j - 2p$ or $i - j$. In the former case the only possibilities are $i + j - 2p = kp$, $k \in \{-1, 0, 1, 2, 3\}$. If $i + j - 2p = \pm p$, then $i - j = 4l$, which implies $2i = 4l \pm 3p$, having no solution. Similarly, if $i + j - 2p = 3p$. The case $i + j = 2p$ and $i + j = 4p$ have no solution for $i \neq j$. Therefore p divides $i - j$, where the only possibility is $i - j = 2p$ (or $i - j = -2p$) or $i - j = p$. If $i - j = \pm p$, then $i + j - 2p = p$, which reduces to a previous case. Thus, we are left with $i - j = \pm 2p$. The proof follows. \square

An important consequence of the previous lemma is that, for p prime, the only irreducible characters which have compatible q -expansions are $\Lambda(i)$, with the lowest conformal weight $h_{i,1}$ and $\Pi(p-i)$, with the lowest conformal weight $h_{2p+i,1}$, $i = 1, \dots, p-1$. Notice also that

$$h_{i,1} < h_{2p+i,1}, \quad 1 \leq i \leq p-1. \quad (6.36)$$

Next result is a consequence of modular properties of irreducible $\mathcal{W}(p)$ characters (cf. [19] for instance). Its proof is well recorded so we omit the proof here (see [19] for instance).

Lemma 6.3. *The vector space spanned by $2p$ -irreducible $\mathcal{W}(p)$ -characters is $2p$ -dimensional. The $SL(2, \mathbb{Z})$ transforms of these characters closes $(3p-1)$ -dimensional $SL(2, \mathbb{Z})$ -module with a basis formed by irreducible characters $\mathrm{tr}_{\Lambda(i)} q^{L(0)-c/24}$, $\mathrm{tr}_{\Pi(i)} q^{L(0)-c/24}$, $1 \leq i \leq p$, and*

$$2\pi i \tau \frac{(\partial\theta)_{p,i}(q)}{\eta(q)}, \quad i = 1, 2, \dots, p-1.$$

Now we are ready to prove the main theorem in this section.

Theorem 6.4. *For every prime p and $i \in \{1, \dots, p-1\}$, $\mathcal{W}(p)$ admits a logarithmic module with a two-dimensional lowest weight subspace of generalized conformal weight $h_{i,1}$.*

Proof. Since the triplet is C_2 -cofinite, every module is $\mathbb{Z}_{\geq 0}$ -gradable and logarithmic (which includes also ordinary modules). Let W be a weak $\mathcal{W}(p)$ -module with a top component $W(0)$. Because of $f_p([\omega]) = 0$, which holds in Zhu's algebra we have $W(0) = \bigoplus_{i=1}^{3p-1} W_{h_{i,1}}(0)$, where $W_{h_{i,1}}(0)$, is $h_{i,1}$ -primary component of $W(0)$, that is $W_{h_{i,1}}$ is annihilated by $(L(0) - h_{i,1})^2$.

From Lemma 6.3 we know that

$$2\pi i \tau \frac{(\partial\theta)_{p,i}(q)}{\eta(q)} \in \tau q^{h_{i,1}-c_{p,1}/24} (b_i + q\mathbb{C}[[q]]), \quad b_i \neq 0, \quad (6.37)$$

is an $SL(2, \mathbb{Z})$ -transform of the set of irreducible characters. Fix an index i . According to the discussion preceding Theorem 6.1, we have

$$2\pi i \tau \frac{(\partial\theta)_{p,i}(q)}{\eta(q)} = \sum_{j=1}^k \mathrm{tr}_{W^j}^{\phi_j} q^{L(0)-c_{p,1}/24}, \quad (6.38)$$

for some pseudotraces $\mathrm{tr}_{W^j}^{\phi_j}$, depending on i , with symmetric maps ϕ_j evaluated on weak modules W^j (if we multiply a pseudotrace with a constant we obtain another pseudotrace). We may assume that on the right-hand side in (6.38) we only have pseudotraces with the coefficients in the τ -expansion being q -series compatible with the left-hand side in (6.38), so let

$$\mathrm{tr}_{W^j}^{\phi_j} q^{L(0)-c_{p,1}/24} \in \sum_{m=0}^r \tau^m q^{\tilde{h}-c_{p,1}/24} \mathbb{C}[[q]],$$

for some \tilde{h} , with $\tilde{h} - h_{i,1} \in \mathbb{Z}$. According to Lemma 6.2 we may assume $\tilde{h} = h_{i,1}$, because there will be no $\mathcal{W}(p)$ -module with generalized conformal weight less than $h_{i,1}$. Of course, because of (6.36), characters of modules with lowest conformal weight $h_{2p+i,1}$ are already contained in $q^{h_{i,1}} \mathbb{C}[[q]]$. Thus we may assume that all W^j have the lowest generalized conformal weights $h_{i,1}$ or $h_{2p+i,1}$. We claim that for at least one W^j , the lowest weight subspace $W^j(0)$ admits at least one non-trivial $L(0)$ -Jordan block of length two of generalized conformal weight $h_{i,1}$. As we already mentioned the property

$$(L(0) - h_{i,1})^2 W^j(0) = 0, \quad i \geq 0,$$

guarantees that $L(0)$ Jordan blocks cannot be of size strictly bigger than 2 (but this does not rule out existence of Jordan blocks of larger size below the top!). Suppose that there is no such W^j , that is suppose that the lowest weight subspace $W^j(0)$ is $L(0)$ -diagonalizable or $W^j(0)$ is trivial. Then from [41], for every j we have

$$\mathrm{tr}_{W^j}^{\phi_j} q^{L(0)-c_{p,1}/24} = \mathrm{tr}_{W^j}^{\phi_j} \sum_{k \geq 0} \frac{(2\pi i \tau)^k (L(0) - L_{ss}(0))^k}{k!} q^{L_{ss}(0)-c_{p,1}/24} \quad (6.39)$$

$$= \sum_{k \geq 0} (2\pi i \tau)^k \mathrm{tr}_{W^j/R_k}^{\tilde{\phi}_k} q^{L_{ss}(0)-c_{p,1}/24}, \quad (6.40)$$

where $R_0 = 0$ and R_k , $k \geq 0$, are certain submodules of W^j . But if $W^j(0)$ is $L(0)$ -diagonalizable then there is no τ -term in (6.39), because $(L(0) - L_{ss}(0))$ annihilate all of $W_i(0)$, so in the (q, τ) -expansion of $\mathrm{tr}_{W^j}^{\phi_j} q^{L(0)-c_{p,1}/24}$ there will be no term of the form $\tau q^{h_{i,1}}$, contradicting to (6.37) and (6.38). \square

Here is an important consequence of the previous theorem.

Theorem 6.5. *The Conjecture 5.8 holds for every prime p .*

Proof. In view of Theorem 5.9 and Conjecture 5.8 we only have to show that each $\mathbb{I}_{h_{i,1}}$ is two-dimensional. Since each v_i is an idempotent and $w_i^2 = 0$, v_i and w_i are not proportional. Thus, it is sufficient to show $w_i \neq 0$, or that w_i acts non-trivially on an $A(\mathcal{W}(p))$ -module. Since $\mathcal{W}(p)$ admits $p - 1$ logarithmic modules, let us call them $\Lambda_2(i)$, $i = 1, \dots, p - 1$, such that $w_i \in A(\mathcal{W}(p))$ acts non-trivially on the top component $\Lambda_2(i)(0)$ the proof follows. \square

The $p = 2$ case of Theorem 6.5 has been verified by Abe [1] by using explicit fermionic construction of logarithmic modules, obtained previously by physicists.

In general, we are able to prove the following result.

Proposition 6.6. *For every $p \geq 2$, the triplet $\mathcal{W}(p)$ admits a logarithmic module of lowest conformal weight $h_{p-1,1}$. Moreover, $\mathbb{I}_{h_{p-1,1}}$ is two-dimensional.*

Proof. The main argument is similar to the one used in the proof of Theorem 6.4 so we omit the details. First observe that $h_{p,1} = \frac{-(p-1)^2}{4p}$ is the smallest conformal weight among all irreducible $\mathcal{W}(p)$ -modules. Since $h_{p-1,1} = \frac{1-(p-1)^2}{4p}$ is not congruent to $h_{p,1} \pmod{\mathbb{Z}}$ for any $p \geq 2$, and $h_{p-1,1} < h_{i,1}$ for other i , then as in Theorem 6.4 it follows that $\mathcal{W}(p)$ admits a logarithmic module with a two-dimensional generalized lowest conformal weight $h_{p-1,1}$. Consequently, $\mathbb{I}_{h_{p-1,1}}$ is two-dimensional. \square

We finish with an expected conjecture

Conjecture 6.7. *There are no logarithmic $\mathcal{W}(p)$ -modules admitting $L(0)$ Jordan blocks of size three or more.*

7. Final remarks and future work

In this section we gather some problems and open question that we shall address in our future publications.

- (i) The problem of constructing logarithmic $\mathcal{W}(p)$ -modules tops the list of our future directions. It seems to us that this problem has not been solved even in the physics literature, except for the $p = 2$ case. We feel that approach in [18] and [40] might be useful for these purposes.
- (ii) Vertex operator algebra $\mathcal{W}(p)$ can be considered as an $\mathfrak{sl}_2 \times \text{Vir}$ -module (cf. [15–17,19,20]). This also follows from our vertex-algebraic approach. Define the following operators acting on $\mathcal{W}(p)$:

$$e = Q, \quad h = \frac{\alpha(0)}{p}.$$

From Theorem 1.1 and Proposition 1.3 it follows that there is a unique operator $f \in \text{End}(\mathcal{W}(p))$ which commutes with the action of the Virasoro algebra such that

$$f e^{-n\alpha} = 0, \quad f Q^j e^{-n\alpha} = -j(j-1-2n)Q^{j-1}e^{-n\alpha}, \quad 1 \leq j \leq 2n.$$

Therefore $\mathcal{W}(p)$ is an $\mathfrak{sl}_2 \times L(c_{p,1}, 0)$ -module and

$$\mathcal{W}(p) = \bigoplus_{n=0}^{\infty} W_{2n+1} \otimes L(c_{p,1}, n^2 p + np - n)$$

where W_{2n+1} is a $(2n+1)$ -dimensional \mathfrak{sl}_2 -module. This implies that the Lie group $PSL(2, \mathbb{C})$ acts on the vertex operator algebra $\mathcal{W}(p)$ as an automorphism group.

Let $\Gamma \subset PSL(2, \mathbb{C})$ be any finite group. Then one can consider the fixed point subalgebra $\mathcal{W}(p)^\Gamma$. We think that it is important and interesting problem to investigate C_2 -cofiniteness of these subalgebras of $\mathcal{W}(p)$.

- (iii) Finally, there is a fair amount of work needed to determine the fusion rules of logarithmic and non-logarithmic $\mathcal{W}(p)$ -modules in the context of vertex algebras. Since the category of $\mathcal{W}(p)$ -modules has a natural braided tensor category structure [8,31,32] it is natural to look for an already existing model for this category. Physicists have provided a beautiful conjecture in this direction: *the tensor category of $\mathcal{W}(p)$ -modules is equivalent to the tensor category of $\overline{U_q(\mathfrak{sl}_2)}$ -modules, $q = e^{\frac{\pi i}{p}}$* . This has been verified for $p = 2$ in [15,17].

Appendix A

A.1. Proof of Theorem 5.1, (iii)

From the Vir-module structure of $\mathcal{W}(p)$ it follows that there exists $a \in \mathcal{U}(\text{Vir}_{\leq -1})$ such that

$$H * F = a.F.$$

Since

$$[H * F] = [H] * [F]$$

in $A(\mathcal{W}(p))$ and

$$\text{wt}(H_{-1}F) = 4p - 2$$

we conclude that there exists a polynomial $q(x) \neq 0$, $\deg(q(x)) \leq p - 1$, such that

$$[H] * [F] = -q([\omega]) * [F].$$

Also, we know that $\Pi(i)$ is an $A(\mathcal{W}(p))$ -module with 2-dimensional top weight subspaces $\Pi(i)(0)$ of lowest conformal weights $h_{i,1}$, $i = 2p, \dots, 3p - 1$. Thus, we obtain the relation

$$[H] * [F]|_{\Pi(i)(0)} = -q([\omega]) * [F]|_{\Pi(i)(0)}. \quad (\text{A.41})$$

It is not hard to see that this relation uniquely determines $q(x)$ and that $\deg(q(x)) = p - 1$ (this will be proven below). As in Section 3, it is convenient to switch to “charge variable” t . Thus, we define

$$H_p(t) = -q\left(\frac{t(t - 2p + 2)}{4p}\right) \in \mathbb{C}[t]. \quad (\text{A.42})$$

Now, $[H]$ acts on $M(1, t)(0)$ as multiplication with $\binom{t}{2p-1}$, and $[\omega]$ acts as multiplication with $\frac{t(t-2p+2)}{4p}$. By using Lagrange interpolation theorem and relations (A.41) and (A.42) it is not hard to prove the following result:

Lemma A.1. *We have*

$$H_p(t) = \left(\frac{1}{(2p-1)!} \prod_{i=2p-1}^{3p-2} (t-i)(t+i-2p+2) \right) \cdot \left(\sum_{i=2p-1}^{3p-2} \frac{(-1)^{p-i}(i!)^2}{(i-2p+1)!^2(p+i)!(3p-i-2)!} \left(\frac{1}{t-i} - \frac{1}{t+i-2p+2} \right) \right).$$

Here are the first few $H_p(t)$ polynomials:

$$H_2(t) = \frac{3}{5}t^2 - \frac{6}{5}t - \frac{4}{5},$$

$$H_3(t) = \frac{5}{84}t^4 - \frac{10}{21}t^3 + \frac{55}{84}t^2 + \frac{25}{21}t + 1,$$

$$H_4(t) = \frac{35}{15444}t^6 - \frac{35}{858}t^5 + \frac{3395}{15444}t^4 - \frac{245}{1287}t^3 - \frac{1897}{3861}t^2 - \frac{3136}{1287}t - \frac{200}{143}$$

and the corresponding $q(x)$ polynomials are then

$$q(x) = -\left(\frac{24}{5}x - \frac{4}{5}\right), \quad \text{for } p = 2,$$

$$q(x) = -\left(\frac{60}{7}x^2 - \frac{25}{7}x + 1\right), \quad \text{for } p = 3,$$

$$q(x) = -\left(\frac{35840}{3861}x^3 - \frac{2240}{351}x^2 + \frac{25088}{3861}x - \frac{200}{143}\right), \quad \text{for } p = 4.$$

As far as we can tell $q(x)$ polynomials do not admit nice factorization so we need a different approach to show that $q(h_{i,1}) \neq 0$. Let

$$H_p(t) = \frac{Pr_p(t)}{(2p-1)!} (S_p(t) + \tilde{S}_p(t)),$$

where

$$\tilde{S}_p(t) = \sum_{i=2p-1}^{3p-2} \frac{-(-1)^{p-i} i!^2}{(t+i-2p+2)(i-2p+1)!^2 (p+i)!(3p-i-2)!},$$

$$S_p(t) = \sum_{i=2p-1}^{3p-2} \frac{(-1)^{p-i} i!^2}{(t-i)(i-2p+1)!^2 (p+i)!(3p-i-2)!}$$

and

$$Pr_p(t) = \prod_{i=2p-1}^{3p-2} (t-i)(t+i-2p+2).$$

Proposition A.2. For every $p \geq 2$,

$$H_p(t) \neq 0, \quad t \in [0, 2p-2] \cap \mathbb{Z}. \quad (\text{A.43})$$

Proof. Notice that $Pr_p(t) \neq 0$ for $t \in \{0, 1, \dots, 2p-2\}$, so we only have to consider

$$A_p(t) = S_p(t) + \tilde{S}_p(t).$$

It is easy to see that

$$A_p(2p-2-t) = A_p(t)$$

for an arbitrary value of parameter t . Thus, relation (A.43) will follow once we prove: $A_p(t) \neq 0$ for $t \in [0, p] \cap \mathbb{Z}$.

It turns out that there is not nice closed formula $A_p(t)$, but $A_p(0)$ and $A_p(1)$ can be computed explicitly. By using standard hypergeometric summation techniques we obtain:

$$A_p(0) = -1/2 \frac{4^p(2p-1) \binom{p-3/2}{p}^2}{\binom{2p-3/2}{p}},$$

$$A_p(1) = -\frac{4^{p-1}(2p-1)(2p^2-1) \binom{p-3/2}{p}^2}{(p^2-1) \binom{2p-3/2}{p}}.$$

Both $A_p(0)$ and $A_p(1)$ are evidently less than zero.

Furthermore, by using straightforward computation we also have a degree two recursion

$$A_p(t) = \frac{(t-1)^2(3p-t)A_p(t-2) + 2(t-p)(t^2-2pt+2p-2p^2)A_p(t-1)}{(t+p)(t+1-2p)^2}.$$

Now it is not hard to see that for $t \in [0, p]$,

$$(t-p)(t^2-2pt+2p-2p^2) \geq 0$$

and of course

$$(t-1)^2(3p-t) \geq 0.$$

Now, inductively it easily follows that $A_p(t) < 0$ for $t \in [0, 2p-2] \cap \mathbb{Z}$. The proof follows. \square

From the previous proposition we get $q(h_{i,1}) \neq 0$.

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